King Fahd University of Petroleum and Minerals

College of Computer Science and Engineering
Information and Computer Science Department

ICS 253: Discrete Structures I
Summer Session 2016-2017
Major Exam #2, Saturday August 5, 2017.
Time: **120** minutes

Name:

ID#:

Instructions:

1. This exam consists of **9** pages, including this page and an additional separate helping sheet, containing **seven** questions.
2. You have to answer all **seven** questions.
3. The exam is closed book and closed notes. No calculators or any helping aides are allowed. Make sure you turn off your mobile phone and keep it in your pocket if you have one.
4. The questions are **NOT equally weighed**. Some questions count for more points than others.
5. The maximum number of points for this exam is **100**.
6. You have exactly **120** minutes to finish the exam.
7. Make sure your answers are **readable**.
8. If there is no space on the front of the page, feel free to use the back of the page. Make sure you indicate this in order for me not to miss grading it.

Question Number	Maximum # of Points	Earned Points
I	**20**	
II	**10**	
III	**20**	
IV	**15**	
V	**10**	
VI	**15**	
VII	**10**	
Total	**100**	

I. (20 points)

1. (5 points) Determine whether these statements are true or false.

   a. $\{\emptyset\} \in \{\emptyset\}$ false

   b. $\{\emptyset\} \in \{\{\emptyset\}\}$ true

   c. $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ true

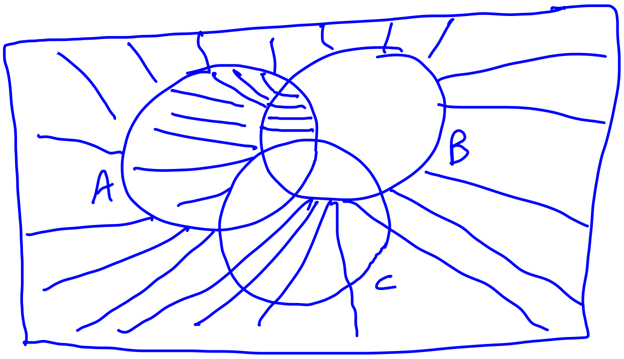
   d. $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$ true

   e. $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$ false

2. (5 points) Find the truth set of $R(x): x \leq x^2$, where the domain is the set of integers.

   $\mathbb{Z}$

3. (5 points) Draw the Venn diagram for $(A \cup \bar{B}) \cap (\bar{A} \cup \bar{C})$ assuming that sets A, B and C are all subsets of the Universal set U.



4. (5 points) Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

$$A_i = (-\frac{1}{i}, i)$$

$$\bigcup_{i=1}^{\infty} A_i = (-1, 1) \cup (-\tfrac{1}{2}, 2) \cup (-\tfrac{1}{3}, 3) \cup \dots$$
$$= (-1, \infty)$$

$$\bigcap_{i=1}^{\infty} A_i = (-1, 1) \cap (-\tfrac{1}{2}, 2) \cap (-\tfrac{1}{3}, 3) \cap (-\tfrac{1}{4}, 4)$$
$$= [0, 1)$$

II. **(10 points)** Prove that the function $f: \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} x+1 & x \in \mathbb{Q} \\ 2x & x \notin \mathbb{Q} \end{cases}$$

is a bijection.

1. f is 1:1

If $f(x) = f(y)$ then

either $x+1 = y+1 \Rightarrow x=y$ OR $2x = 2y \Rightarrow x=y$.

The last case to consider is $f(x)=f(y)$ where $x+1 = 2y$.

i.e. $y = \frac{x+1}{2}$. However this cannot happen since y is not rational, and having $y = \frac{x+1}{2}$ with $x \in \mathbb{Q}$ makes $y \in \mathbb{Q}$

2. f is onto.

Let $y \in \mathbb{R}$. Then $y \in \mathbb{Q}$ or $y \notin \mathbb{Q}$

if $y \in \mathbb{Q}$, then having $y = x+1 \Rightarrow x = y-1$.

if $y \notin \mathbb{Q}$, then having $y = 2x \Rightarrow x = \frac{y}{2}$.

from these conditions, we can conclude that

f is onto.

III. (20 points)

1. (10 points) Compute the following summation:

$$S = \sum_{i=0}^{100} \sum_{j=10}^{20} (i)(j)(2^{i-1})$$

$$S = \sum_{i=0}^{100} i\,2^{i-1} \left(\sum_{j=10}^{20} j \right)$$

$$= \sum_{i=0}^{100} i\,2^{i-1} \left(\sum_{j=1}^{20} j - \sum_{j=1}^{9} j \right)$$

$$(210 - 45) = 165$$

$$= \sum_{i=0}^{100} i\,2^{i-1} \left(\frac{20(21)}{2} - \frac{9(10)}{2} \right)$$

$$= \frac{165}{2} \sum_{i=0}^{100} i\,2^{i}$$

$$= \frac{165}{2} \left(\frac{100(2^{102}) - 100(2^{101}) - 2^{101} + 2}{1^2} \right)$$

$$= \frac{165}{2} \left(100(2^{102}) - 101(2^{101}) + 2 \right)$$

2. (10 points) Give an example of two uncountable sets A and B such that $A \cap B$ is

   a. Empty.

   $A = [0, 1]$ $B = [2, 3]$

   b. Finite but not empty.

   $A = [0, 1]$ $B = [1, 2]$

   c. Countably infinite.

   $A = [0, 1] \cup \mathbb{Z}$ $B = [1, 2] \cup \mathbb{Z}$

   d. Uncountable.

   $A = [0, 2]$ $B = A$

IV. (15 points)

1. (5 points) Provide a simple formula that generates the terms of an integer sequence that begins with the list

$$2, 3, 7, 25, 121, 721, 5041, \ldots$$

and then find the following term.

$f(n) = n! + 1$

next term is 40321.

2. (5 points) Rewrite the following expression using the summation notation Σ.

$$1 + (1 + 4) + (1 + 4 + 7) + \big(1 + 4 + 7 + \cdots + (3n - 2)\big)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{i} (3j - 2)$$

3. (5 points) A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with *n* cars made in the *n*th month.

   a. Set up a recurrence relation for the number of cars produced in the first *n* months by this factory.

$a_1 = 1$

$a_n = n + a_{n-1}$ $\quad n \geq 2.$

$\quad = a_{n-1} + n$

   b. How many cars are produced in the first year?

$a_{12} = a_{11} + 12$

$\quad = a_{10} + 11 + 12$

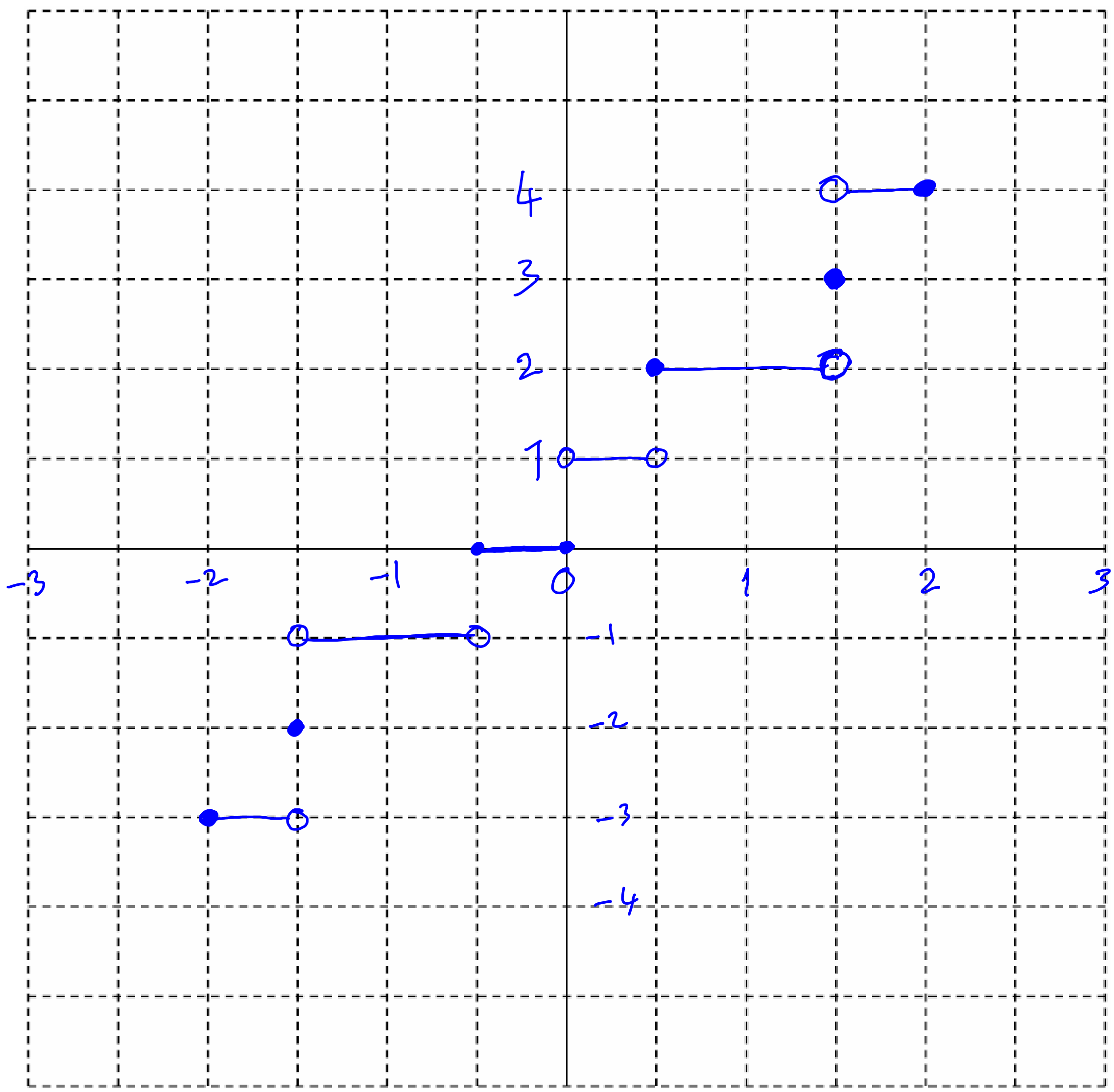
$\quad \vdots$

$\quad = \sum_{i=1}^{12} i = \frac{12(13)}{2} = 6(13)$

$\quad = 78$

V. **(10 points)** Draw the graph of the function

$$f(x) = \left\lceil \frac{2x}{3} \right\rceil + \left\lfloor x + \frac{1}{2} \right\rfloor \qquad \text{where } -2 \le x \le 2$$



①

$\left\lceil \frac{2x}{3} \right\rceil = -1 \Leftrightarrow -2 < \frac{2x}{3} \le -1$

$-3 < x \le -\frac{3}{2}$

$= 0 \Leftrightarrow -1 < \frac{2x}{3} \le 0$

$-\frac{3}{2} < x \le 0$

$= 1 \Leftrightarrow 0 < \frac{2x}{3} \le 1$

$0 < x \le \frac{3}{2}$

$= 2 \Leftrightarrow 1 < \frac{2x}{3} \le 2$

$\frac{3}{2} < x \le 3$

②

$\left\lfloor x + \frac{1}{2} \right\rfloor = -2 \Leftrightarrow -2 \le x + \frac{1}{2} < -1$

$-2.5 \le x < -1.5$

$= -1 \Leftrightarrow -1 \le x + \frac{1}{2} < 0$

$-1.5 \le x < 0.5$

$= 0 \Leftrightarrow 0 \le x + \frac{1}{2} < 1$

$-0.5 \le x < 0.5$

$= 1 \Leftrightarrow 0.5 \le x < 1.5$

$= 2 \Leftrightarrow 1.5 \le x < 2.5$

Interval	①	②	①+②
$[-2, -\frac{3}{2})$	-1	-2	-3
-1.5	-1	-1	-2
$(-1.5, -0.5)$	0	-1	-1
$[-0.5, 0]$	0	0	0
$(0, 0.5)$	1	0	1
$[0.5, 1.5)$	1	1	2

Interval	①	②	+
1.5	1	2	3
$(1.5, 2]$	2	2	4

VI. **(15 points)** Use mathematical induction to prove that

$$3 * 6 + 6 * 9 + \cdots + 3n(3n+3) = 3n(n+1)(n+2)$$

Basis: $P(1)$: $3*6 = 18 \stackrel{?}{=} 3(1)(2)(3) = 3(6) = 18$ ✓

Inductive step: Assume that the result holds for n, i.e.

$$\sum_{i=1}^{n} 3i(3i+3) = 3n(n+1)(n+2)$$

To show that $\sum_{i=1}^{n+1} 3i(3i+3) = 3(n+1)(n+2)(n+3)$.

Now, By Induction hypothesis,
$$\sum_{i=1}^{n+1} 3i(3i+3) = 3n(n+1)(n+2) + 3(n+1)[3(n+1)+3]$$
$$= 3(n+1)[n(n+2) + 3n+6]$$
$$= 3(n+1)[n(n+2) + 3(n+2)]$$
$$= 3(n+1)[(n+2)(n+3)]$$
$$= 3(n+1)(n+2)(n+3)$$

VII. **(10 points)** Prove that the following statements are equivalent about the integer n:

a. n^2 is odd.
b. $1 - n$ is even.
c. n^3 is odd.

$a \to b$:

Assume that $1-n$ is odd.
i.e. $1-n = 2k+1$
$n = -2k \iff n = 2(-k)$ with $-k \in \mathbb{Z}$.
$\therefore n^2 = (2(-k))^2 = 4k^2$ is even.

$b \to c$: Assume $1-n$ is even

$1-n = 2k, \quad k \in \mathbb{Z}$.

$n = 1-2k$

$n^2 = 1 - 4k + 4k^2$

$n^3 = 1 - 6k + 12k^2 - 8k^3$

$= 2(-3k + 6k^2 - 4k^3) + 1$ which is odd since $-3k + 6k^2 - 4k^3 \in \mathbb{Z}$.

$c \to a$:

Assume that n^2 is even

Then, $n^2 = 2k \quad k \in \mathbb{Z}$.

$n^3 = n(2k) = 2(nk)$

but $nk \in \mathbb{Z}$

$\therefore n^3$ is even

Some Useful Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad , \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad , \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1} \text{ where } a \neq 1 \quad , \quad \sum_{i=0}^{\infty} a^i = \frac{1}{1-a} \text{ where } |a|<1,$$

$$\sum_{i=1}^{\infty} i a^{i-1} = \frac{1}{(1-a)^2} \text{ where } |a|<1$$

$p \to (p \vee q)$	Addition	$[\neg q \wedge (p \to q)] \to \neg p$	Modus Tollens
$(p \wedge q) \to p$	Simplification	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$((p) \wedge (q)) \to (p \wedge q)$	Conjunction	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
$[p \wedge (p \to q)] \to q$	Modus Ponens	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

Some Useful Sequences	
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, …
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, …
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, 14641, …
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, …
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147, …
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916800
f_n Fibonacci	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

$A \cap U = A$ $A \cup \Phi = A$	Identity Laws	$A \cup U = U$ $A \cap \Phi = \Phi$	Domination Laws
$A \cap A = A$ $A \cup A = A$	Idempotent Laws	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$\overline{(\bar{A})} = A$	Complementation Law	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
$\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$	De Morgan's Laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive Laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws	$A \cup \bar{A} = U$ $A \cap \bar{A} = \Phi$	Complement Laws